Asymptotic Solution of the Dirac Equation*

S. I. RUBINOW†
Stevens Institute of Technology, Hoboken, New Jersey

AND

JOSEPH B. KELLER
Courant Institute of Mathematical Sciences,
New York University, New York, New York
(Received 26 February 1963)

The WKB method is applied to solve the Dirac equation and the modified Dirac equation appropriate to a spin-$\frac{1}{2}$ particle with an anomalous magnetic moment. The solution consists of a phase factor multiplied by a spinor amplitude which is a power series in Planck's constant. The phase is a solution of the Hamilton-Jacobi equation of relativistic mechanics for a spinless particle without electric or magnetic moments. Each term in the spinor amplitude satisfies an ordinary differential equation along the relativistic trajectories. The equation for the leading amplitude yields an equation for the polarization four-vector which is identical with that derived classically by Bargmann, Michel, and Telegdi. It also yields the law of conservation of probability in a tube of trajectories. In addition, it gives rise to an equation for a supplementary phase factor. By using the classical Hamilton-Jacobi function, the law of probability conservation, the polarization four-vector and the supplementary phase factor, the leading term in the solution of the Dirac or modified Dirac equation can be constructed. This solution should be useful when the wavelength of the particle is small compared to the characteristic distance associated with the electromagnetic potential through which the particle moves. When applied to the bound states of a particle without an anomalous moment in a spherically symmetric electrostatic potential, it yields the same results as are usually obtained by separation of variables and use of the ordinary WKB method. The advantage of the present method is that it applies equally well to nonseparable problems.

1. INTRODUCTION

AN attempt was made by Pauli[1] to solve the Dirac equation for a particle in an electromagnetic field using the WKB method. He sought a solution ψ of the form

$$\psi \sim e^{i\hbar^{-1}S} \sum_{n=0}^{\infty} (-i\hbar)^n a_n, \tag{1}$$

where $\hbar = h/2\pi$. Upon inserting (1) into the Dirac equation and equating to zero the coefficient of each power of $\hbar$, he obtained equations for the scalar function S and the spinor functions a_n. S was found to satisfy the Hamilton-Jacobi equation of relativistic mechanics for a spinless charged particle without electric or magnetic moments, so it can be determined by means of the particle trajectories. However, a_n was found to satisfy a system of partial differential equations which Pauli was unable to solve in general. We have succeeded in solving them by reducing them to ordinary differential equations along the particle trajectories. In the same way, we have solved the modified Dirac equation appropriate to a particle with an anomalous magnetic moment.

The equation for a_0 leads to an equation for the precession of the polarization four-vector which was derived classically by Bargmann, Michel, and Telegdi.[2] Quantum mechanically it holds exactly for a particle in a homogeneous field. We obtain this equation for inhomogeneous fields as well, but only to lowest order in $\hbar$. Thus, according to our analysis the spin and moments do not affect the trajectories, but the moments affect the precession of the spin, which varies along the trajectory in accordance with the appropriate covariant equation of motion. Another consequence of the equation for a_0 is conservation of probability in a tube of trajectories. A third consequence is an equation for an additional phase which depends upon the velocity and polarization. By employing the classical equations for the trajectories and for the polarization, we can construct a_0 and S. Upon using them in (1) and neglecting higher terms, we obtain an approximate wave function constructed from classical quantities. This approximate wave function may prove useful in solving problems in which the particle wavelength is small compared to the characteristic lengths associated with the electromagnetic field, which is the nondimensional meaning of $\hbar$ being small.

We have constructed the approximate wave function for the ordinary Dirac equation for a particle in a uniform magnetic field and for bound states of a particle in a spherically symmetric electrostatic potential. In each case the condition that ψ must be single-valued leads, in a known way,[3] to the appropriate

* This research was supported by the National Science Foundation under Grant No. 19671 and by the U. S. Office of Naval Research.
† Also, Courant Institute of Mathematical Sciences, New York University, New York, New York.
[1] W. Pauli, Helv. Phys. Acta 5, 179 (1932).
[2] V. Bargmann, L. Michel, and V. L. Telegdi, Phys. Rev. Letters 2, 435 (1959).
[3] J. B. Keller, Ann. Phys. (N. Y.) 4, 180 (1958); J. B. Keller and S. I. Rubinow, *ibid.* 9, 24 (1960).

quantum conditions which determine the energy levels. In both cases the problems can be solved by separation of variables and application of the usual WKB method to the resulting separated equations. The results obtained in this way coincide with ours. Therefore, we do not present these examples.

De Broglie[4] has criticized Pauli's procedure, which we have followed, because the trajectories to which it leads are unaffected by the particle's electric and magnetic moments. In the next section we analyze this criticism and show to what extent it is valid. At the same time, we point out the limitations on the validity of Pauli's procedure.

2. DE BROGLIE'S CRITICISM OF PAULI'S PROCEDURE

De Broglie's criticism of Pauli's procedure is based on the fact that the "limit" of quantum mechanics as $\hbar$ tends to zero is classical mechanics. Since electric and magnetic moments are classical concepts, they should remain in the classical "limit" and should affect the trajectories. In Pauli's procedure they do remain but they do not affect the trajectories, which depend only upon the charge and mass of the particle. This would appear to be a shortcoming of his procedure. However, the moments of an electron are proportional to $\hbar$ and, therefore, they too vanish in the classical limit. (The equation for the precession of the magnetic moment involves only the ratio of the magnetic moment to the spin, from which $\hbar$ cancels.) This vanishing of the moments seems to justify Pauli's procedure and to invalidate de Broglie's objection. Nevertheless, as we shall now show, his objection is valid despite the fact that Pauli's procedure is correct. The explanation of this paradoxical statement is that Pauli's procedure yields the correct result in inhomogeneous field regions and at fixed finite distances from them, but not at distances of the order $\hbar^{-1}$ from them. The correct result at such distances can only be obtained by taking account of de Broglie's objection, and permitting the moments to affect the trajectories.

To clarify the above explanation, let us consider the classical motion through an inhomogeneous field of finite extent of an electron with electric and magnetic moments proportional to $\hbar$. The angular deviation of the trajectory produced by the inhomogeneity is proportional to the moments of the electron, and thus proportional to $\hbar$. Therefore, as $\hbar$ tends to zero, the trajectory at every point approaches the trajectory of an electron with moments. However, the approach is not uniform at infinity. At a distance of order $\hbar^{-1}$ along the trajectory from the inhomogeneity, the lateral deflection of the trajectory is the product of the angular deviation of order $\hbar$ and the distance of order $\hbar^{-1}$, which product is of order unity (i.e., independent of $\hbar$). Therefore, to obtain a description of the trajectories which is valid everywhere, including the neighborhood of infinity, it is necessary to include the effect of the moments even though the moments themselves vanish with $\hbar$. However, a description which is valid at any finite point, but is not uniformly valid at infinity, can be obtained by ignoring the moments.

Let us now reformulate the preceding explanation analytically. We seek the asymptotic expansion of the wave function $\psi(x,\hbar)$ as $\hbar$ tends to zero. If x is fixed, it is given by the WKB expansion (1) employed by Pauli. However, if $x=\hbar^{-1}x'$, where x' is fixed, then the asymptotic exansion of $\psi(\hbar^{-1}x',\hbar)$ as $\hbar$ tends to zero is not given by (1). It is given by a different expansion which would meet de Broglie's objection in that the trajectories would be affected by the moments. The first term in such an expansion could presumably be obtained from Schiller's[5] approximate solution of the squared Dirac equation. We shall not determine that expansion in this paper.

These considerations may become more understandable by the examination of a familiar situation in which a similar phenomenon occurs, namely the occurrence of a shadow behind a sphere of radius a illuminated by a plane wave of wavelength λ. We know that when λ is much smaller than a, the shadow is essentially a circular cylinder of radius a, but that it disappears at about the distance a^2/λ behind the sphere. Therefore, if we wish to determine the behavior of the shadow in the geometrical optics limit, in which λ/a tends to zero, we must specify whether we want it at a fixed distance or at a distance a^2/λ behind the sphere. At a fixed distance there is a shadow whose cross section is a circle while at a distance a^2/λ there is no shadow. A uniform description would describe the circular shadow and its gradual disappearance as the distance increases.

This example illustrates the main point involved in reconciling the viewpoints of Pauli and de Broglie. It is that a function of x and a parameter $\hbar$ may have more than one asymptotic expansion with respect to $\hbar$ around $\hbar=0$, and they are valid in different domains of x space. The validity domain of one expansion may even include that of another. In seeking an expansion, it is necessary to specify the domain in which it is to be valid. By ignoring this point, Pauli failed to realize that his expansion was not uniformly valid at infinity and de Broglie failed to see that what he wanted was an expansion which was uniformly valid everywhere, including infinity. The domain of validity of this expansion would include that of the WKB expansion considered by Pauli, but the expansion would be more difficult to determine.

3. FORMULATION

The modified Dirac equation for the wave function of an electron with an anomalous magnetic moment

[4] L. de Broglie, *La Theorie des Particules de Spin 1/2* (Gauthier-Villars, Paris, 1952), pp. 132, 128.

[5] R. Schiller, Phys. Rev. **128**, 1402 (1962).

$-(\frac{1}{2}g-1)e\hbar/mc$ may be written in the form

$$[\gamma_\mu(\hbar\partial_\mu+iec^{-1}A_\mu)+mc-(\tfrac{1}{2}g-1)\times(ie\hbar/2mc^2)F_{\mu\nu}\sigma_{\mu\nu}]\psi=0. \quad (2)$$

The notation in (2) is the following:

$$\partial_\mu=\partial/\partial x_\mu,\quad x_\mu=(x,y,z,ict),\quad A_\mu=(A_x,A_y,A_z,i\Phi),\quad (3)$$
$$\sigma_{\mu\nu}=\tfrac{1}{2}(\gamma_\mu\gamma_\nu-\gamma_\nu\gamma_\mu),\quad F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu.$$

Here x, y, z, and t are the Cartesian space coordinates and the time, respectively; c is the velocity of light; A_x, A_y, A_z and Φ are the three components of the electromagnetic vector potential and the scalar potential, respectively; $F_{\mu\nu}$ is the electromagnetic field tensor, the index μ ranges from 1 to 4 and is to be summed over when it is repeated in a given term; m is the rest mass of the electron, $(-e)$ is its charge, and g is its gyromagnetic ratio. The wave function ψ is a four-component column vector and the γ_μ are the 4 by 4 matrices

$$\gamma_j=i\begin{bmatrix}0 & -\sigma_j\\ \sigma_j & 0\end{bmatrix},\quad j=1,2,3;\quad \gamma_4=\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}. \quad (4)$$

The σ_j are the 2 by 2 Pauli spin matrices. When $g=2$, (2) becomes the usual Dirac equation.

We seek a solution of (2) of the form (1). Therefore, we insert (1) into (2) and equate to zero the coefficient of each power of $\hbar$. This yields the equations

$$(i\pi_\mu\gamma_\mu+mc)a_0=0, \quad (5)$$

$$(i\pi_\mu\gamma_\mu+mc)a_n=-i\gamma_\mu\partial_\mu a_{n-1}-(\tfrac{1}{2}g-1)(e/2mc^2)\times F_{\mu\nu}\sigma_{\mu\nu}a_{n-1},\quad n=1,2,\cdots. \quad (6)$$

Here π_μ is defined by

$$\pi_\mu=\partial_\mu S+ec^{-1}A_\mu. \quad (7)$$

Equation (5) is a system of linear algebraic equations for a_0. It has a nontrivial solution only if the coefficient matrix has a vanishing determinant,

$$det\ (i\pi_\mu\gamma_\mu+mc)=0. \quad (8)$$

Upon evaluation of this determinant we obtain

$$\pi_\mu\pi_\mu+m^2c^2=0. \quad (9)$$

Equation (9) is just the single-particle relativistic Hamilton-Jacobi equation for the function S describing the motion of a spinless particle without electric or magnetic moments.

When S satisfies (9) the matrix in (5) is found to have rank 2. Therefore, (5) has $4-2=2$ linearly independent solutions. Denoting them by B_1 and B_2 we find

$$B_1=\begin{bmatrix}u\\0\\v\\w_+\end{bmatrix},\quad B_2=\begin{bmatrix}0\\u\\w_-\\-v\end{bmatrix},$$

$$B_i^\dagger B_j=-2i\pi_4(mc-i\pi_4)\delta_{ij};\quad i,j=1,2. \quad (10)$$

Here u, v, and $w_\pm$ are defined by

$$u=mc-i\pi_4,\quad v=\pi_3,\quad w_\pm=\pi_1\pm i\pi_2. \quad (11)$$

We shall later utilize two linearly independent left null vectors of the matrix in (5). These are easily found to be the Pauli adjoints of B_1 and B_2, namely,

$$\bar{B}_1=B_1^\dagger\gamma_4=(u^*,0,-v^*,-w_+^*),$$
$$\bar{B}_2=B_2^\dagger\gamma_4=(0,u^*,-w_-^*,v^*). \quad (12)$$

In (10) and (12) the † designates the Hermitian adjoint; i.e., the complex conjugate of the transpose, the overbar denotes the Pauli adjoint, and the asterisk denotes the complex conjugate.

In terms of B_1 and B_2 we may write a_n as

$$a_0=\alpha_{01}B_1+\alpha_{02}B_2,$$
$$a_n=\alpha_{n1}B_1+\alpha_{n2}B_2+b_n,\quad n=1,2,\cdots. \quad (13)$$

Here b_n is a particular solution of (6) while α_{n1} and α_{n2} are scalar functions which are so far undetermined. To determine them we consider Eqs. (6). These are inhomogeneous linear algebraic equations for a_n. They have solutions only if the right side is orthogonal to all the solutions of the transposed homogeneous equations. But these latter solutions are linear combinations of $\bar{B}_1$ and $\bar{B}_2$. Therefore, the conditions for solvability are obtained by multiplying (6) on the left by $\bar{B}_1$ and $\bar{B}_2$, which are left null vectors of the coefficient matrix. The resulting solvability conditions are

$$\bar{B}_1\gamma_\mu\partial_\mu a_{n-1}-(\tfrac{1}{2}g-1)(ie/2mc^2)\bar{B}_1F_{\mu\nu}\sigma_{\mu\nu}a_{n-1}=0,$$
$$\bar{B}_2\gamma_\mu\partial_\mu a_{n-1}-(\tfrac{1}{2}g-1)(ie/2mc^2)\bar{B}_2F_{\mu\nu}\sigma_{\mu\nu}a_{n-1}=0, \quad (14)$$
$$n=1,2,\cdots.$$

These equations were obtained by Pauli[1] for $g=2$. He did not see how to solve them in general. Instead, after deducing one consequence, related to Eq. (25) below, he turned to a special case. We shall show how to solve (14), in general.

4. REDUCTION TO ORDINARY DIFFERENTIAL EQUATIONS

Let us now substitute for a_n from (13) into (14). Then (14) becomes for $n=0,1,\cdots$

$$(\bar{B}_1\gamma_\mu B_1)\partial_\mu\alpha_{n1}+(\bar{B}_1\gamma_\mu\partial_\mu B_1)\alpha_{n1}+(\bar{B}_1\gamma_\mu B_2)\partial_\mu\alpha_{n2}+(\bar{B}_1\gamma_\mu\partial_\mu B_2)\alpha_{n2}-(\tfrac{1}{2}g-1)(ie/2mc^2)\times\bar{B}_1F_{\mu\nu}\sigma_{\mu\nu}(\alpha_{n1}B_1+\alpha_{n2}B_2+b_n)=-\bar{B}_1\gamma_\mu\partial_\mu b_n. \quad (15)$$

$$(\bar{B}_2\gamma_\mu B_1)\partial_\mu\alpha_{n1}+(\bar{B}_2\gamma_\mu\partial_\mu B_1)\alpha_{n1}+(\bar{B}_2\gamma_\mu B_2)\partial_\mu\alpha_{n2}+(\bar{B}_2\gamma_\mu\partial_\mu B_2)\alpha_{n2}-(\tfrac{1}{2}g-1)(ie/2mc^2)\times\bar{B}_2F_{\mu\nu}\sigma_{\mu\nu}(\alpha_{n1}B_1+\alpha_{n2}B_2+b_n)=-\bar{B}_2\gamma_\mu\partial_\mu b_n. \quad (16)$$

Here we have introduced $b_0=0$ to enable us to write the equations for $n=0$ together with those for $n\neq 0$. For each n, Eqs. (15) and (16) are a pair of linear first order partial differential equations for α_{n1} and α_{n2}. We shall now show that they can be reduced to ordinary differential equations along the particle paths associated with (9). The reduction is based upon the following algebraic theorem.

Theorem

$$\bar{B}_j\gamma_\mu B_k = -2i(mc-i\pi_4)\pi_\mu\delta_{jk}, \quad j, k=1, 2. \tag{17}$$

This theorem can be proved by direct computation. A more algebraic proof is given in Appendix I.

We now use (17) in (15) and (16). Then after multiplying these equations by $[-2i(mc-i\pi_4)]^{-1}$ they can be written as

$$\begin{aligned} D\alpha_{n1}+c_{11}\alpha_{n1}+c_{12}\alpha_{n2} &= [2i(mc-i\pi_4)]^{-1}\bar{B}_1[\gamma_\mu\partial_\mu-(\tfrac{1}{2}g-1) \\ &\quad\times(ie/2mc^2)F_{\mu\nu}\sigma_{\mu\nu}]b_n, \end{aligned} \tag{18}$$

$$\begin{aligned} D\alpha_{n2}+c_{21}\alpha_{n1}+c_{22}\alpha_{n2} &= [2i(mc-i\pi_4)]^{-1}\bar{B}_2[\gamma_\mu\partial_\mu-(\tfrac{1}{2}g-1) \\ &\quad\times(ie/2mc^2)F_{\mu\nu}\sigma_{\mu\nu}]b_n. \end{aligned} \tag{19}$$

Here D and the c_{ij} are defined by

$$D=\pi_\mu\partial_\mu, \tag{20}$$

$$\begin{aligned} c_{ij} &= [-2i(mc-i\pi_4)]^{-1}\bar{B}_i[\gamma_\mu\partial_\mu-(\tfrac{1}{2}g-1) \\ &\quad\times(ie/2mc^2)F_{\mu\nu}\sigma_{\mu\nu}]B_j, \quad i, j=1, 2. \end{aligned} \tag{21}$$

Since D is a directional derivative in the direction π_μ, (18) and (19) are ordinary differential equations along the curves to which π_μ is tangential. These curves are the relativistic trajectories of a particle in the potential A_μ. An incorrect version of (18) and (19) for $g=2$ was given by de Broglie.[4]

Let us now consider the case $n=0$, set $\alpha_{01}=\alpha_1$, $\alpha_{02}=\alpha_2$, and assume that S is real. Then D is real and an immediate consequence of (18) and (19) for $n=0$ is the equation

$$\begin{aligned} &D(\alpha_1\alpha_1^*+\alpha_2\alpha_2^*)+\alpha_1\alpha_1^*(c_{11}+c_{11}^*)+\alpha_2\alpha_2^*(c_{22}+c_{22}^*) \\ &\quad+\alpha_1\alpha_2^*(c_{21}+c_{12}^*)+\alpha_1^*\alpha_2(c_{12}+c_{21}^*)=0. \end{aligned} \tag{22}$$

It follows from (21) and the definitions of the B_i that

$$c_{11}+c_{11}^*=c_{22}+c_{22}^*=\partial_\mu\pi_\mu+D\ln(mc-i\pi_4), \tag{23}$$

$$c_{21}+c_{12}^*=0. \tag{24}$$

Thus, (22) becomes

$$D(\alpha_1\alpha_1^*+\alpha_2\alpha_2^*)+(c_{11}+c_{11}^*)(\alpha_1\alpha_1^*+\alpha_2\alpha_2^*)=0. \tag{25}$$

To solve (25) we write $D=md/d\tau$ where τ is a parameter along a trajectory which can easily be recognized as the proper time. Then, the solution of (25) is

$$\begin{aligned} \alpha_1\alpha_1^*+\alpha_2\alpha_2^* &= (\alpha_1\alpha_1^*+\alpha_2\alpha_2^*)_{\tau_0}\exp\left(-\frac{1}{m}\int_{\tau_0}^{\tau}(c_{11}+c_{11}^*)d\tau\right), \\ &= (\alpha_1\alpha_1^*+\alpha_2\alpha_2^*)_{\tau_0}\frac{mc-i\pi_4(\tau)}{mc-i\pi_4(\tau_0)}\exp\left(-\frac{1}{m}\int_{\tau_0}^{\tau}\partial_\mu\pi_\mu d\tau\right). \end{aligned} \tag{26}$$

The integral in (26) is evaluated in Appendix II and is found to be

$$\frac{1}{m}\int_{\tau_0}^{\tau}\partial_\mu\pi_\mu d\tau=\ln\frac{d\sigma(\tau)}{d\sigma(\tau_0)}. \tag{27}$$

The quantity $d\sigma(\tau)$ is the three dimensional cross-section of a narrow tube of trajectories containing the one under consideration, evaluated at the point τ on this trajectory. $d\sigma(\tau_0)$ is the corresponding cross section at the point τ_0. Actually $d\sigma(\tau)/d\sigma(\tau_0)$ is the limit of the cross-section ratio as $d\sigma(\tau_0)$ tends to zero. When (27) is used in (26), it finally becomes

$$\alpha_1\alpha_1^*+\alpha_2\alpha_2^*=(\alpha_1\alpha_1^*+\alpha_2\alpha_2^*)_{\tau_0}\frac{[mc-i\pi_4(\tau)]d\sigma(\tau_0)}{[mc-i\pi_4(\tau_0)]d\sigma(\tau)}. \tag{28}$$

This equation expresses conservation of probability in a tube of trajectories.

We can now use the above results to simplify Eqs. (18) and (19) for α_1 and α_2. To this end we introduce the two component vector

$$\beta=\begin{pmatrix}\beta_1\\ \beta_2\end{pmatrix}$$

defined by

$$\beta=\exp\left(\frac{1}{2m}\int_{\tau_0}^{\tau}\partial_\mu\pi_\mu d\tau\right)\begin{pmatrix}\alpha_1\\ \alpha_2\end{pmatrix}. \tag{29}$$

Then (18) and (19) with $n=0$, and (25) yield for β the equation

$$D\beta+M\beta=0. \tag{30}$$

In (30), M is a square matrix of order two with elements

$$M_{ij}=c_{ij}-\tfrac{1}{2}\partial_\mu\pi_\mu\delta_{ij}, \tag{31}$$

The determination of the solution a_0 has now been reduced to the problem of finding the vector β satisfying (30). Then α_1 and α_2 are given by (29) and a_0 by (13), which becomes upon using (27)

$$a_0=\left[\frac{d\sigma(\tau_0)}{d\sigma(\tau)}\right]^{1/2}(\beta_1B_1+\beta_2B_2). \tag{32}$$

The spinors B_1 and B_2 are given by (10) and (11).

5. PRECESSION OF THE POLARIZATION FOUR-VECTOR

We shall now show that (30) implies the covariant equation of motion of the polarization four-vector along a particle trajectory. Furthermore, a solution of that equation can be used to construct the solution β of (30). To this end we utilize the vector $\boldsymbol{\sigma}$, whose components are the Pauli spin matrices of order two. We multiply (30) from the left by $\beta^\dagger\boldsymbol{\sigma}$ and obtain

$$\beta^\dagger\boldsymbol{\sigma}D\beta+\beta^\dagger\boldsymbol{\sigma}M\beta=0. \tag{33}$$

The Hermitian adjoint of (33) is

$$(D\beta^\dagger)\boldsymbol{\sigma}\beta+\beta^\dagger M^\dagger\boldsymbol{\sigma}\beta=0. \tag{34}$$

Addition of (33) and (34) yields

$$D(\beta^\dagger\boldsymbol{\sigma}\beta)+\beta^\dagger(\boldsymbol{\sigma}M+M^\dagger\boldsymbol{\sigma})\beta=0. \tag{35}$$

This equation can be concisely written in terms of the vector $\boldsymbol{\Sigma}$ defined by

$$\boldsymbol{\Sigma}=\beta^\dagger\boldsymbol{\sigma}\beta. \tag{36}$$

From (36), the components and length Σ of $\boldsymbol{\Sigma}$ are

$$\Sigma_x=\beta_1\beta_2{}^*+\beta_1{}^*\beta_2, \tag{37}$$

$$\Sigma_y=i(\beta_1\beta_2{}^*-\beta_1{}^*\beta_2), \tag{38}$$

$$\Sigma_z=\beta_1\beta_1{}^*-\beta_2\beta_2{}^*, \tag{39}$$

$$\Sigma=(\Sigma_x{}^2+\Sigma_y{}^2+\Sigma_z{}^2)^{1/2}=\beta_1\beta_1{}^*+\beta_2\beta_2{}^*. \tag{40}$$

The matrix M, given by (31), involves c_{ij} defined by (21), which in turn involves B_1 and B_2 given by (10) and (11). When M is evaluated from these equations, it can be expressed in terms of the electromagnetic tensor $F_{\mu\nu}$, the components of which are the components of the electric field $\mathbf{E}$ and the magnetic field $\mathbf{H}$. Thus,

$$M_{11}=\frac{ei}{2c}\left\{H_z+\frac{(\mathbf{E}\times\boldsymbol{\pi})_z+i\mathbf{E}\cdot\boldsymbol{\pi}}{mc-i\pi_4}+\left(\frac{g}{2}-1\right)\times\left(H_z+\frac{(\mathbf{E}\times\boldsymbol{\pi})_z}{mc}+\frac{[\boldsymbol{\pi}\times(\mathbf{H}\times\boldsymbol{\pi})]_z}{mc(mc-i\pi_4)}\right)\right\}, \tag{41}$$

$$M_{12}=\frac{ei}{2c}\left\{H_x-iH_y+\frac{(\mathbf{E}\times\boldsymbol{\pi})_x-i(\mathbf{E}\times\boldsymbol{\pi})_y}{mc-i\pi_4}+\left(\frac{g}{2}-1\right)\times\left(H_x-iH_y+\frac{(\mathbf{E}\times\boldsymbol{\pi})_x-i(\mathbf{E}\times\boldsymbol{\pi})_y}{mc}+\frac{[\boldsymbol{\pi}\times(\mathbf{H}\times\boldsymbol{\pi})]_x-i[\boldsymbol{\pi}\times(\mathbf{H}\times\boldsymbol{\pi})]_y}{mc(mc-i\pi_4)}\right)\right\}, \tag{42}$$

$$M_{21}=\frac{ei}{2c}\left\{H_x+iH_y+\frac{(\mathbf{E}\times\boldsymbol{\pi})_x+i(\mathbf{E}\times\boldsymbol{\pi})_y}{mc-i\pi_4}+\left(\frac{g}{2}-1\right)\times\left(H_x+iH_y+\frac{(\mathbf{E}\times\boldsymbol{\pi})_x+i(\mathbf{E}\times\boldsymbol{\pi})_y}{mc}+\frac{[\boldsymbol{\pi}\times(\mathbf{H}\times\boldsymbol{\pi})]_x+i[\boldsymbol{\pi}\times(\mathbf{H}\times\boldsymbol{\pi})]_y}{mc(mc-i\pi_4)}\right)\right\}, \tag{43}$$

$$M_{22}=-\frac{ei}{2c}\left\{H_z+\frac{(\mathbf{E}\times\boldsymbol{\pi})_z-i\mathbf{E}\cdot\boldsymbol{\pi}}{mc-i\pi_4}+\left(\frac{g}{2}-1\right)\times\left(H_z+\frac{(\mathbf{E}\times\boldsymbol{\pi})_z}{mc}+\frac{[\boldsymbol{\pi}\times(\mathbf{H}\times\boldsymbol{\pi})]_z}{mc(mc-i\pi_4)}\right)\right\}. \tag{44}$$

Here $\boldsymbol{\pi}=(\pi_1,\pi_2,\pi_3)$. Upon using (36)–(44) in (35) we obtain the following equation for $\boldsymbol{\Sigma}$:

$$D\boldsymbol{\Sigma}=-\frac{e}{c}\left\{\boldsymbol{\Sigma}\times\mathbf{H}+\frac{\boldsymbol{\Sigma}\times(\mathbf{E}\times\boldsymbol{\pi})-\boldsymbol{\Sigma}(\mathbf{E}\cdot\boldsymbol{\pi})}{mc-i\pi_4}+\left(\frac{g}{2}-1\right)\times\left(\boldsymbol{\Sigma}\times\mathbf{H}+\frac{\boldsymbol{\Sigma}\times(\mathbf{E}\times\boldsymbol{\pi})}{mc}+\frac{\boldsymbol{\Sigma}\times[\boldsymbol{\pi}\times(\mathbf{H}\times\boldsymbol{\pi})]}{mc(mc-i\pi_4)}\right)\right\}. \tag{45}$$

We shall now introduce the polarization four-vector T_μ and show that it can be expressed in terms of $\boldsymbol{\Sigma}$. To do so we recall that the spin current density is given by $\bar{\psi}i\gamma_5\gamma_\mu\psi$ with $\gamma_5=\gamma_1\gamma_2\gamma_3\gamma_4$. Therefore, the flux of this current through a tube of trajectories of cross section $d\sigma(\tau)$ is $\bar{\psi}i\gamma_5\gamma_\mu\psi d\sigma(\tau)$. Let us compute this flux by using the expansion (1) for ψ and retaining only the first term so that $\psi\sim e^{i\hbar^{-1}S}a_0$. We shall call this flux the polarization four-vector T_μ multiplied by the fixed cross section $d\sigma(\tau_0)$ so that

$$T_\mu=e^{-i\hbar^{-1}S}\bar{a}_0i\gamma_5\gamma_\mu a_0e^{i\hbar^{-1}S}\frac{d\sigma(\tau)}{d\sigma(\tau_0)}. \tag{46}$$

Upon using (32) for a_0 and the definition (36) of $\boldsymbol{\Sigma}$, we find

$$\mathbf{T}=2mc(mc-i\pi_4)\boldsymbol{\Sigma}+2(\boldsymbol{\pi}\cdot\boldsymbol{\Sigma})\boldsymbol{\pi}, \tag{47}$$

$$T_4=2i(mc-i\pi_4)\boldsymbol{\pi}\cdot\boldsymbol{\Sigma}. \tag{48}$$

In (47) the vector $\mathbf{T}=(T_1,T_2,T_3)$.

Equations (47) and (48) show that the polarization four-vector T_μ is expressible in terms of $\boldsymbol{\Sigma}$. Since $\boldsymbol{\Sigma}$ satisfies (45), these equations will enable us to get an equation satisfied by T_μ. To do so we shall first solve (47) for $\boldsymbol{\Sigma}$ in terms of $\mathbf{T}$. Thus, multiplying (47) by $\boldsymbol{\pi}\cdot$ and solving for $\boldsymbol{\pi}\cdot\boldsymbol{\Sigma}$ yields

$$\boldsymbol{\pi}\cdot\boldsymbol{\Sigma}=[-2i\pi_4(mc-i\pi_4)]^{-1}\boldsymbol{\pi}\cdot\mathbf{T}. \tag{49}$$

Now (47) and (49) lead to

$$\boldsymbol{\Sigma}=[2mc(mc-i\pi_4)]^{-1}\times\{\mathbf{T}+[i\pi_4(mc-i\pi_4)]^{-1}(\boldsymbol{\pi}\cdot\mathbf{T})\boldsymbol{\pi}\}. \tag{50}$$

To obtain the equation for T_μ, we need to know the equations for a trajectory

$$D\pi_\mu=-(e/c)F_{\mu\nu}\pi_\nu=-(e/c)(-i\pi_4\mathbf{E}+(\boldsymbol{\pi}\times\mathbf{H});+i\mathbf{E}\cdot\boldsymbol{\pi}). \tag{51}$$

This is the Lorentz force equation, which follows directly from the Hamilton-Jacobi equation. It is convenient to utilize more common notation by recognizing that the four-velocity $dx_\mu/d\tau$ is just

$$\frac{dx_\mu}{d\tau}=\gamma\left(\frac{\mathbf{v}}{c},i\right)=\frac{\pi_\mu}{mc}, \tag{52}$$

where

$$\gamma=(1-v^2/c^2)^{-1/2}. \tag{53}$$

Then equations (45), (50), and (51) may be rewritten as

$$\frac{d\boldsymbol{\Sigma}}{d\tau}=-\frac{e}{mc}\Big\{\boldsymbol{\Sigma}\times\mathbf{H}+\frac{1}{mc(1+\gamma)}[\boldsymbol{\Sigma}\times(\mathbf{E}\times\boldsymbol{\pi})-\boldsymbol{\Sigma}(\mathbf{E}\cdot\boldsymbol{\pi})]+\Big(\frac{g}{2}-1\Big)\Big(\boldsymbol{\Sigma}\times\mathbf{H}+\frac{1}{mc}\boldsymbol{\Sigma}\times(\mathbf{E}\times\boldsymbol{\pi})+\frac{1}{m^2c^2(1+\gamma)}\boldsymbol{\Sigma}\times[\boldsymbol{\pi}\times(\mathbf{H}\times\boldsymbol{\pi})]\Big)\Big\},\quad(54)$$

$$\boldsymbol{\Sigma}=[2m^2c^2(1+\gamma)]^{-1}\Big[\mathbf{T}-\frac{1}{m^2c^2\gamma(1+\gamma)}(\mathbf{T}\cdot\boldsymbol{\pi})\boldsymbol{\pi}\Big],\quad(55)$$

$$\frac{d\boldsymbol{\pi}}{d\tau}=-e\Big[\gamma\mathbf{E}-\frac{\mathbf{H}\times\boldsymbol{\pi}}{mc}\Big],\quad(56)$$

and

$$\frac{d\gamma}{d\tau}=-\frac{e}{m^2c^2}\mathbf{E}\cdot\boldsymbol{\pi}.\quad(57)$$

Substituting (55), (56), and (57) into (54), we obtain the equation

$$\frac{d\mathbf{T}}{d\tau}-\frac{1}{m^2c^2\gamma(1+\gamma)}\Big(\boldsymbol{\pi}\cdot\frac{d\mathbf{T}}{d\tau}\Big)\boldsymbol{\pi}=-\frac{e}{mc}\Big\{\mathbf{T}\times\mathbf{H}+\frac{(\mathbf{T}\cdot\boldsymbol{\pi})\mathbf{E}}{mc\gamma}-\Big[(\mathbf{T}\times\mathbf{H})\cdot\boldsymbol{\pi}+\frac{(\mathbf{T}\cdot\boldsymbol{\pi})(\mathbf{E}\cdot\boldsymbol{\pi})}{mc\gamma}\Big]\frac{\boldsymbol{\pi}}{m^2c^2\gamma(1+\gamma)}+\Big(\frac{g}{2}-1\Big)\times\Big(\mathbf{T}\times\mathbf{H}+\frac{\mathbf{T}\cdot\boldsymbol{\pi}}{mc\gamma}\mathbf{E}-\Big[\mathbf{T}\cdot\mathbf{E}-\frac{(\mathbf{T}\times\mathbf{H})\cdot\boldsymbol{\pi}}{mc(1+\gamma)}-\frac{(\mathbf{T}\cdot\boldsymbol{\pi})(\mathbf{E}\cdot\boldsymbol{\pi})}{m^2c^2\gamma(1+\gamma)}\Big]\frac{\boldsymbol{\pi}}{mc}\Big)\Big\}.\quad(58)$$

Multiplying (58) by $\boldsymbol{\pi}\cdot$, we find that

$$\boldsymbol{\pi}\cdot\frac{d\mathbf{T}}{d\tau}=-\frac{e}{mc}\Big\{(\mathbf{T}\times\mathbf{H})\cdot\boldsymbol{\pi}+\frac{(\mathbf{T}\cdot\boldsymbol{\pi})\mathbf{E}\cdot\boldsymbol{\pi}}{mc\gamma}+\Big(\frac{g}{2}-1\Big)\gamma^2\Big[(\mathbf{T}\times\mathbf{H})\cdot\boldsymbol{\pi}+\frac{(\mathbf{T}\cdot\boldsymbol{\pi})\mathbf{E}\cdot\boldsymbol{\pi}}{mc\gamma}-(\mathbf{T}\cdot\mathbf{E})\frac{mc(1+\gamma)(\gamma-1)}{\gamma}\Big]\Big\}.\quad(59)$$

By substituting (59) back into Eq. (58), the equation for $d\mathbf{T}/d\tau$ becomes

$$\frac{d\mathbf{T}}{d\tau}=-\frac{e}{mc}\Big\{\frac{g}{2}\Big[\mathbf{T}\times\mathbf{H}+\frac{\mathbf{T}\cdot\boldsymbol{\pi}}{mc\gamma}\mathbf{E}\Big]+\Big(\frac{g}{2}-1\Big)\gamma^2\Big[-\frac{\mathbf{T}\cdot\mathbf{E}}{mc\gamma}+\frac{(\mathbf{T}\times\mathbf{H})\cdot\boldsymbol{\pi}}{(mc\gamma)^2}+\frac{(\mathbf{T}\cdot\boldsymbol{\pi})(\mathbf{E}\cdot\boldsymbol{\pi})}{(mc\gamma)^3}\Big]\boldsymbol{\pi}\Big\}.\quad(60)$$

In a similar manner, the equation satisfied by T_4 is found to be

$$\frac{dT_4}{d\tau}=-\frac{ei}{mc}\Big\{\frac{g}{2}\mathbf{T}\cdot\mathbf{E}+\Big(\frac{g}{2}-1\Big)\gamma^2\times\Big[-\mathbf{T}\cdot\mathbf{E}+\frac{(\mathbf{T}\times\mathbf{H})\cdot\boldsymbol{\pi}}{mc\gamma}+\frac{(\mathbf{T}\cdot\boldsymbol{\pi})(\mathbf{E}\cdot\boldsymbol{\pi})}{(mc\gamma)^2}\Big]\Big\}.\quad(61)$$

Equations (60) and (61) together are the covariant equations of motion for the polarization four-vector, $T_\mu=(\mathbf{T},T_4)$, of an electron moving in an arbitrary electromagnetic field.[2]

6. DETERMINATION OF β

To solve for β in terms of $\mathbf{T}$, we use (50) to get $\boldsymbol{\Sigma}$ in terms of $\mathbf{T}$ and then (37)–(40) to get β in terms of $\boldsymbol{\Sigma}$ and hence of $\mathbf{T}$. Thus, we find from Eqs. (37)–(40) that β may be expressed in terms of $\boldsymbol{\Sigma}$ as follows:

$$\beta=\frac{e^{i\theta}}{\sqrt{2}}\begin{pmatrix}(\Sigma+\Sigma_z)^{1/2}\exp\Big[-\frac{i}{2}\tan^{-1}\Big(\frac{\Sigma_y}{\Sigma_x}\Big)\Big]\\(\Sigma-\Sigma_z)^{1/2}\exp\Big[\frac{i}{2}\tan^{-1}\Big(\frac{\Sigma_y}{\Sigma_x}\Big)\Big]\end{pmatrix}.\quad(62)$$

Inasmuch as four real functions are required to completely determine β, it is seen that the three components of $\boldsymbol{\Sigma}$ leave a phase function θ to be determined. The equation for θ may be found by substituting (62) into (30) and multiplying by $\beta^\dagger$. It follows that

$$2\Sigma D\theta=\Sigma_z D[\tan^{-1}(\Sigma_y/\Sigma_x)]+i\beta^\dagger(M-M^\dagger)\beta.\quad(63)$$

By direct calculation it is found that

$$D\Big[\tan^{-1}\Big(\frac{\Sigma_y}{\Sigma_x}\Big)\Big]=\frac{(\boldsymbol{\Sigma}\times D\boldsymbol{\Sigma})_z}{\Sigma_x^2+\Sigma_y^2}\quad(64)$$

and

$$i\beta^\dagger(M-M^\dagger)\beta=\frac{e}{c}\boldsymbol{\Sigma}\cdot\Big\{\mathbf{H}+\frac{(\mathbf{E}\times\boldsymbol{\pi})}{mc-i\pi_4}+\Big(\frac{g}{2}-1\Big)\times\Big[\mathbf{H}+\frac{\mathbf{E}\times\boldsymbol{\pi}}{mc}+\frac{\boldsymbol{\pi}\times(\mathbf{H}\times\boldsymbol{\pi})}{mc(mc-i\pi_4)}\Big]\Big\}.\quad(65)$$

Hence, with the use of (45), it follows from (63) that

$$\frac{d\theta}{d\tau}=-\frac{e}{2mc}\frac{\Sigma}{\Sigma_x^2+\Sigma_y^2}(\Sigma_x,\Sigma_y,0)\cdot\left\{\mathbf{H}+\frac{(\mathbf{E}\times\boldsymbol{\pi})}{mc(1+\gamma)}+\left(\frac{g}{2}-1\right)\left[\mathbf{H}+\frac{(\mathbf{E}\times\boldsymbol{\pi})}{mc}+\frac{\boldsymbol{\pi}\times(\mathbf{H}\times\boldsymbol{\pi})}{m^2c^2(1+\gamma)}\right]\right\}. \quad (66)$$

The solution may be written immediately as

$$\theta=\theta_0-\frac{e}{2mc}\int_{\tau_0}^{\tau}\frac{\Sigma}{\Sigma_x^2+\Sigma_y^2}(\Sigma_x,\Sigma_y,0)\cdot\left\{\frac{g}{2}\left[\mathbf{H}+\frac{(\mathbf{E}\times\boldsymbol{\pi})}{mc(1+\gamma)}\right]+\left(\frac{g}{2}-1\right)[mc(1+\gamma)]^{-1}\times\left[\gamma(\mathbf{E}\times\boldsymbol{\pi})+\frac{\boldsymbol{\pi}\times(\mathbf{H}\times\boldsymbol{\pi})}{mc}\right]\right\}d\tau. \quad (67)$$

Here θ_0 is the value of θ at $\tau=\tau_0$.

7. CONCLUSION

We have now completed the determination of the zero-order solution a_0 since (67) determines θ, (62) determines β and then (32) yields a_0. Upon combining these results in (1), we obtain

$$\psi\sim e^{-i\hbar^{-1}S}a_0=\frac{e^{-i\hbar^{-1}S+i\theta}}{\sqrt{2}}\left[\frac{d\sigma(\tau_0)}{d\sigma(\tau)}\right]^{1/2}[(\Sigma+\Sigma_z)^{1/2}\times\exp[-\tfrac{1}{2}i\tan^{-1}(\Sigma_y/\Sigma_x)]B_1+(\Sigma-\Sigma_z)^{1/2}\times\exp[\tfrac{1}{2}i\tan^{-1}(\Sigma_y/\Sigma_x)]B_2]. \quad (68)$$

In (68), S is the relativistic Hamilton-Jacobi function, θ is a phase factor given by (67), $d\sigma(\tau_0)/d\sigma(\tau)$ is the ratio of the cross sections of an infinitesimal tube of rays at τ_0 and τ, where τ is proper time along a trajectory, $\boldsymbol{\Sigma}$ is a vector given in terms of the polarization four vector T_μ by (55) and B_1 and B_2 are spinors given by (10). Although (68) may seem complicated, it actually involves only classical quantities associated with a classical trajectory. For that reason, it should be useful in the approximate solution of problems since classical quantities can be found by solving ordinary differential equations. Thus, the solution (68) is obtainable from the solution of ordinary differential equations even for nonseparable problems.

To determine further terms in the expansion of ψ, the ordinary differential equations (18) and (19) must be solved for $n>0$. They can be analyzed in a way similar to that employed above for $n=0$ but we shall not do so.

The result (68) and some of the calculations leading to it can be simplified somewhat by the introduction of a polarization basis for the solutions of the free Dirac equation (5) rather than the basis B_1, B_2 which we have employed. Our basis refers to an axis fixed in the laboratory coordinate system, while the polarization basis is a frame which refers to an axis that rotates with the polarization T_μ of the particle. In order to define this basis we first observe from the definition of T_μ that $T_\mu\pi_\mu=0$. Furthermore, from (60) and (61) it follows that $(d/d\tau)T_\mu T_\mu=0$ so we may normalize T_μ by setting $T_\mu T_\mu=1$. Then from (68) we find that $\psi^\dagger\psi\sim\frac{1}{2}\gamma d\sigma(\tau_0)/d\sigma(\tau)$. We also find from (68) that $i\gamma_5\gamma_\mu T_\mu\psi=\psi$. Thus, (68) is an eigenvector of the operator $i\gamma_5\gamma_\mu T_\mu$. Let us define the two vectors u_+ and u_- to be solutions of the free Dirac equation (5) normalized by the condition $u_\pm^\dagger u_\pm=1$ and satisfying

$$i\gamma_5\gamma_\mu T_\mu u_\pm=\pm u_\pm. \quad (69)$$

These vectors $u_\pm$ form the polarization basis and in terms of them (68) becomes

$$\psi\sim e^{-i\hbar^{-1}S+i\theta}\left[\frac{d\sigma(\tau_0)}{d\sigma(\tau)}\right]^{1/2}(\tfrac{1}{2}\gamma)^{1/2}u_+. \quad (70)$$

The definition of $u_\pm$ does not fix their phase, and (70) is true only if the phase of u_+ is the same as that of the bracketed expression in (68). However, if the phase of u_+ included the phase θ, then θ could be omitted from (70). Of course, it would then remain present in u_+. The use of the polarization basis together with some results of Bouchiat and Michel[6] pertaining to it, can also simplify some of our other equations and calculations.

APPENDIX I: PROOF OF THEOREM

To prove the theorem of Sec. 3 we consider n Hermitian matrices M_μ and n real scalars p_μ, $\mu=1,\cdots n$. Let G be the Hermitian matrix defined by

$$G=\sum_{\mu=1}^{n}p_\mu M_\mu. \quad (A1)$$

Let λ be a multiple eigenvalue of G and $B_1,\cdots,B_q$ a set of corresponding orthormal eigenvectors which are differentiable functions of p_μ. Then

$$B_j^\dagger B_k=\delta_{jk}, \quad (A2)$$

$$GB_k=\lambda B_k. \quad (A3)$$

Let us now differentiate (A3) with respect to p_μ to obtain

$$M_\mu B_k+G\frac{\partial B_k}{\partial p_\mu}=\frac{\partial\lambda}{\partial p_\mu}B_k+\lambda\frac{\partial B_k}{\partial p_\mu}. \quad (A4)$$

Multiplication of (A4) on the left by B_j^+, the use of (A3), and the fact that G is Hermitian yield

$$B_j^\dagger M_\mu B_k=\frac{\partial\lambda}{\partial p_\mu}\delta_{jk}. \quad (A5)$$

[6] C. Bouchiat and L. Michel, Nucl. Phys. 5, 416 (1958).

This is the desired result. The above derivation is essentially due to R. M. Lewis.

To apply (A5) we first multiply (5) from the left by γ_4 and obtain

$$(i\pi_\mu\gamma_4\gamma_\mu+mc\gamma_4)a_0=0. \tag{A6}$$

The matrices $i\gamma_4\gamma_j$ ($j=1, 2, 3$), $\gamma_4\gamma_4$ and γ_4 are Hermitian so they may be chosen as the M_μ ($\mu=1, \cdots, 5$) of the above analysis. When S is real, as we now assume, π_j ($j=1, 2, 3$), $i\pi_4$ and mc are real and may be chosen to be the p_μ ($\mu=1, \cdots, 5$) above. Then the matrix in (A6) is G and B_1 and B_2 given by (10) are two orthogonal eigenvectors of G corresponding to the eigenvalue $\lambda=0$. They can be normalized by dividing them by an appropriate factor which can be found from (10). Now (A5) applies and yields, when the normalization of B_j is taken into account,

$$B_j^\dagger i\gamma_4\gamma_\mu B_k=(\partial\lambda/\partial p_\mu)[-2i\pi_4(mc-i\pi_4)]\delta_{jk}, \quad \mu=1, 2, 3 \tag{A7}$$

$$B_j^\dagger\gamma_4\gamma_4 B_k=-2i\pi_4(mc-i\pi_4)\delta_{jk}. \tag{A8}$$

In (A8) we have used the fact thet $\gamma_4\gamma_4$ is the identity, so the left side is just $B_j^\dagger B_k$ which is given by (10). Since $B_j^\dagger\gamma_4=\bar{B}_j$, the left side of (A8) and $-i$ times the left sides of (A7) are the left sides of the theorem, Eq. (17). The right side of (A8) is the same as that of (17), which proves (17) for $\mu=4$. To prove it for $\mu=1, 2, 3$ one may evaluate the left side for $j=k=1$ and $\mu=1, 2, 3$ to show that the right side has the value given by (17).

APPENDIX II: EVALUATION OF AN INTEGRAL

We shall now evaluate the integral in (26), the value of which is given in (27). To do so we consider a four-dimensional volume V bounded by an infinitesimal tube of trajectories and by two three-dimensional orthogonal cross sections of the tube. Let the proper time at these ends of the tube be τ_0 and τ on one of the trajectories. Then by Gauss' theorem

$$\int_V \partial_\mu\pi_\mu dV=\int_S \pi_\mu\cdot dS_\mu=|\pi_\mu(\tau)|A(\tau) -|\pi_\mu(\tau_0)|A(\tau_0). \tag{B1}$$

The surface integral in (B1) is evaluated by noting that π_μ is parallel to the sides of the tube so that $\pi_\mu\cdot dS_\mu=0$ there whereas π_μ is normal to the ends of the tube so $\pi_\mu\cdot dS_\mu=|\pi_\mu(\tau)|A(\tau)$ on one end and $=-|\pi_\mu(\tau_0)| \times A(\tau_0)$, on the other. Here dS_μ denotes an element of the surface directed normally and $A(\tau)$ is the cross-sectional "area" of the tube at τ.

Let us rewrite (B1) by noting that $dV=A(\tau)icd\tau$ and that $|\pi_\mu(\tau)|=imc$. Then (B1) becomes

$$ic\int_{\tau_0}^{\tau} \partial_\mu\pi_\mu A(\tau)d\tau=imc[A(\tau)-A(\tau_0)]. \tag{B2}$$

Differentiating (B2) with respect to τ and dividing the result by $A(\tau)$ yields

$$\partial_\mu\pi_\mu=md\ln A(\tau)/d\tau. \tag{B3}$$

Integrating (B3) yields

$$\frac{1}{m}\int_{\tau_0}^{\tau} \partial_\mu\pi_\mu d\tau=\ln[A(\tau)/A(\tau_0)]. \tag{B4}$$

If we denote the cross-sectional area by $d\sigma(\tau)$ instead of $A(\tau)$, (B4) is exactly (27).